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Fisher equation with density-dependent diffusion: special solutions

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Abstract

A special family of solutions explicit in the space and time variables is found for the Fisher equation with density-dependent diffusion. The connection with the known travelling wave solution and the initial conditions from which that evolves is also shown.

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The Fisher equation (FE) [1] is the simplest example of a nonlinear reaction diffusion equation. This equation has been used to describe the growth and dispersal of a variety of biological organisms [2, 3] and, in particular, the rate at which these organisms invade and become established in new settings. The analytical theory of the FE is based on the seminal study of Kolmogorov *et al* [4] who proved that a compact initial condition evolves at long times as a travelling wave. Therefore, although explicit spacetime solutions of the FE are not known, travelling wave solutions have been found for both the FE and generalized FEs [3, 5].

The nonlinearity in the FE occurs through the growth term alone with dispersal treated as classical diffusion. The case where dispersal increases locally as population increases, i.e. individuals seek to avoid crowding, is also of considerable interest [6–9]. In this case, both the diffusion and growth terms in the generalized FE are nonlinear and this results in a travelling wave solution that is qualitatively different from the FE solution. The latter results in a continuous wave profile while the former results in a discontinuous wave profile with the density identically zero ahead of the wavefront. This might be expected since a similar behaviour occurs for the simpler porous media equation [2, 7] which also has a nonlinear diffusion term (but no growth term); a discontinuous wave profile also results when the dispersal process is a correlated random walk [10].

Here we consider the modified FE with nonlinear diffusion. The existing travelling wave solutions of this equation [7–9] are based on a phase-plane analysis combined with a numerical study of the phase portraits. We take a less restricted approach and directly find a family of special solutions in the space and time variables that include the travelling wave solution as a special case. This allows us to identify the family of compact initial conditions that evolves as a discontinuous travelling wave with specified wave speed. In terms of dimensionless space and time variables the equation we consider is

$$n_t = (n^2)_{xx} + n\,(1-n) \tag{1}$$

which is equivalent to the same equation with n_{xx}^2 replaced by $(nn_x)_x$ and x rescaled. Equation (1) can be rewritten as

$$w_T = (w^2)_{xx} - w^2 \tag{2}$$

where $n = \mathrm{e}^t w \geqslant 0$ and $\mathrm{d}T = \mathrm{e}^t\,\mathrm{d}t$. Following the substitutions $X = \mathrm{e}^{2x}$ and $W = \mathrm{e}^{x/2}w$ equation (2) is rewritten as

$$X^{7/4} W_T = 4(W^2)_{XX} \tag{3}$$

and for initial conditions symmetric about $x = 0$ we need not explicitly concern ourselves with $x < 0$ where the solution is given by the reflection of $n(x,t)$ for positive x. Equation (3) admits the family of solutions

$$W = \frac{1}{T + c_1}\left[1 - \frac{c_2 X^{1/4}}{(T + c_1)^{1/2}}\right] \tag{4}$$

where c_1 and c_2 are constants. Returning to the original variable n and space and time coordinates x, t we then have

$$n(x,t) = \frac{\mathrm{e}^t}{(\mathrm{e}^t + c_1 - 1)}\left[1 - \frac{c_2\,\mathrm{e}^{x/2}}{(\mathrm{e}^t + c_1 - 1)^{1/2}}\right] \tag{5}$$

for all values of $x, t \geqslant 0$ for which $n(x,t) \geqslant 0$ and $n(x,t) = 0$ otherwise.

The condition that $n(x,t) \geqslant 0$ imposes certain restrictions on the constants c_1 and c_2 since the initial condition that generates the above solution is $n(x,0) = c_1^{-1}\big[1 - \big(c_2/c_1^{1/2}\big)\,\mathrm{e}^{x/2}\big]$ for $x \leqslant 2\ln\big(c_1^{1/2}/c_2\big) \equiv x^*$ and $n(x,0) = 0, x > x^*$. This implies that $c_1^{1/2} > c_2 > 0$. With these restrictions, the above family of compact initial conditions evolves as the compact family of solutions given by equation (5). Further, for the special allowable case $c_1 = 1$, the solution is a travelling wave with wave speed $v = 1$, the minimum wave speed found previously [7, 8] by considering long time solutions of the form $n(x,t) \to n(x - vt) \equiv n(z)$ and analysis of the trajectories in the n, n_z phase plane.

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